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The initial oscillatory flow past a circular cylinder

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Abstract. This paper presents results obtained from an initial approximation for the flow around a circular cylinder in two-dimensional oscillating flow. The analysis is developed in terms of the scalar vorticity and stream function. An expansion in powers of time from the start of the motion is obtained using an exact analysis which extends the results of boundary-layer theory by taking into account corrections for finite Reynolds number. The time development of the physical properties of the flow are determined both by means of analytical expressions and by an accurate numerical procedure. The surface pressure, drag and surface vorticity are calculated and various estimates of the time of separation and the distance moved in this time are obtained. The phenomenon of steady streaming is not considered in this paper since the time of validity of the expansions is small. The agreement between the analytical and numerical results at small times is excellent.

1. Introduction

Oscillating flow past a fixed circular cylinder has received a great deal of attention especially for the prediction of the loads on engineering structures such as ocean pipelines or risers, offshore-platform supports, bridge piers or smoke stacks. This classical fluid mechanics problem has previously been studied extensively (Stuart [1], Wang [2], Riley [3], Stuart [4], Davidson & Riley [5], Vasantha & Riley [6]). In these studies perturbation methods based on unsteady boundary-layer theory was used. It seems that there is no work on finite Reynolds number perturbations based on the full Navier-Stokes equations. The essential purpose of the present paper is to give the details of the initial oscillatory flow past a circular cylinder for high but finite values of the Reynolds number when $\alpha = O(1)$, where α is the Strouhal number. The present analysis adopts basically the same type of perturbation method as that used by Collins & Dennis [7] for solving the initial flow (non-oscillating) past a circular cylinder.

2. Governing equations and method of solution

At time $t = 0$, the viscous incompressible fluid surrounding an infinitely long circular cylinder suddenly starts to oscillate at right angles to the axis of symmetry of the cylinder. Unidirectional oscillations of the flow are represented by the velocity $U \cos \omega t$, where ω is the frequency of the oscillations and t is the time. In practice, modified polar co-ordinates (ξ, θ) are used, where $\xi = \log(r/a)$, a is the radius of the cylinder, and the origin is taken at the centre of the cylinder.

The motion is two-dimensional and may be described in terms of the usual two simultaneous equations satisfied by the stream function and the scalar vorticity. Dimensionless functions ψ and ζ are used, related to the dimensional stream function and vorticity $\psi^* = Ua\psi$,

$\zeta^* = -U\zeta/a$. The dimensionless radial and transverse components of velocity (u, v) obtained by dividing the corresponding dimensional components by U are then given by

$$u = \mathrm{e}^{-\xi}\frac{\partial\psi}{\partial\theta}, \quad v = -\mathrm{e}^{-\xi}\frac{\partial\psi}{\partial\xi}, \tag{2.1}$$

and the function ζ is defined by

$$\zeta = \mathrm{e}^{-\xi}\left(\frac{\partial u}{\partial\theta} - \frac{\partial v}{\partial\xi} - v\right). \tag{2.2}$$

The equations of motion can be expressed as the two equations

$$\mathrm{e}^{2\xi}\frac{\partial\xi}{\partial\tau} = \frac{2}{R}\left(\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\theta^2}\right) - \frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial\xi} + \frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\theta}, \tag{2.3}$$

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\theta^2} = \mathrm{e}^{2\xi}\zeta. \tag{2.4}$$

Here $\tau = Ut/a$ and R is the Reynolds number defined by $R = 2Ua/\nu$, where ν is the coefficient of kinematic viscosity.

Equations (2.3) and (2.4) are those considered by Collins & Dennis [7] in the case of sudden translation of a circular cylinder without oscillation. Here the oscillation of the flow enters through the Strouhal number $\alpha = a\omega/U$ in the boundary conditions, which may be stated as

$$\psi = \frac{\partial\psi}{\partial\xi} = 0 \quad \text{when } \xi = 0, \tag{2.5}$$

$$\mathrm{e}^{-\xi}\frac{\partial\psi}{\partial\xi} \to (\cos\alpha\tau)\sin\theta, \quad \mathrm{e}^{-\xi}\frac{\partial\psi}{\partial\theta} \to (\cos\alpha\tau)\cos\theta \quad \text{as } \xi \to \infty. \tag{2.6}$$

The set of conditions (2.5) and (2.6) must be satisfied for all $\tau > 0$ and for all θ such that $0 \leqslant \theta \leqslant \pi$, and moreover, the flow will be assumed to remain symmetrical about the direction of the motion of the flow. Then both functions ψ and ζ are anti-symmetrical about $\theta = 0$ and $\theta = \pi$ and, in particular,

$$\psi(\xi,\theta) = \zeta(\xi,\theta) = 0 \quad \text{when } \theta = 0, \theta = \pi. \tag{2.7}$$

In the present analysis the calculations are carried out on the basis of the method of solution adopted by Collins & Dennis [7] in which the functions ψ and ζ were expressed in the form of the series

$$\psi(\xi,\theta,\tau) = \sum_{n=1}^{\infty} f_n(\xi,\tau)\sin n\theta, \tag{2.8}$$

$$\zeta(\xi,\theta,\tau) = \sum_{n=1}^{\infty} g_n(\xi,\tau)\sin n\theta \tag{2.9}$$

to determine the initial flow in the boundary layer mainly by analytical methods for small values of τ. We shall use the same coordinates and transformations appropriate to the initial boundary layer as those employed by Collins & Dennis [7]. Thus, the basic governing equations remain unaltered and will be summarized briefly in the next section, but the boundary conditions are different and give rise to the essential differences in the flow.

The boundary conditions follow from (2.5) and (2.6). From (2.5) we find that

$$f_n = \frac{\partial f_n}{\partial \xi} = 0 \quad \text{when } \xi = 0, \tag{2.10}$$

for all n. As a consequence of the condition (2.6) we must also have that, for all n,

$$g_n(\xi, \tau) \to 0 \quad \text{as } \xi \to \infty. \tag{2.11}$$

Finally, the condition (2.6) implies that

$$e^{-\xi}\frac{\partial f_n}{\partial \xi} \to (\cos\alpha\tau)\delta_{n,1}, \quad e^{-\xi} f_n \to (\cos\alpha\tau)\delta_{n,1}, \quad \text{as } \xi \to \infty, \tag{2.12}$$

where $\delta_{m,n}$ is the Kronecker delta symbol defined by

$$\delta_{m,n} = 1 \text{ if } m = n, \quad \delta_{m,n} = 0 \text{ if } m \neq n. \tag{2.13}$$

It may now be shown, following Collins & Dennis [7], that (2.10) and (2.12) can be used to deduce from (2.4) a further set of conditions of global type, namely

$$\int_0^\infty e^{(2-n)\xi} g_n(\xi, \tau)\, d\xi = 2(\cos\alpha\tau)\delta_{n,1} \tag{2.14}$$

where $\delta_{n,1}$ has the significance in (2.13). It can be shown that the conditions (2.10), (2.11) and (2.14) are sufficient to solve the problem, and that, if they are satisfied and $g_n(\xi, \tau)$ is assumed to be such that $e^{2\xi} g_n(\xi, \tau)$ is bounded for all n as $\xi \to \infty$, then the flow is automatically adjusted to satisfy the external stream condition (2.6). The functions $g_n(\xi, \tau)$ can be verified to satisfy the necessary condition a posteriori.

3. Determination of the initial flow

The initial flow is governed by the usual boundary-layer theory in which a layer of thickness $(\tau/R)^{1/2}$ surrounds the cylinder following the sudden start. We therefore introduce variables appropriate to this layer defined by

$$\xi = kx, \quad k = 2(2\tau/R)^{1/2}, \tag{3.1}$$

and then transform all the appropriate equations using (3.1) together with the scalings of variables

$$f_n = kF_n, \quad g_n = \frac{G_n}{k}. \tag{3.2}$$

The equation (2.4) then gives rise to the set of equations

$$\frac{\partial^2 F_n}{\partial x^2} - n^2 k^2 F_n = e^{2kx} G_n, \tag{3.3}$$

while equation (2.3) gives the set

$$4\tau\frac{\partial G_n}{\partial\tau} = \mathrm{e}^{-2kx}\frac{\partial^2 G_n}{\partial x^2} + (2x + 4n\tau F_{2n}\,\mathrm{e}^{-2kx})\frac{\partial G_n}{\partial x} + \left[2 + \mathrm{e}^{-2kx}\left(2n\tau\frac{\partial F_{2n}}{\partial x} - n^2k^2\right)\right]G_n + 4\tau\,\mathrm{e}^{-2kx}S_n^*, \qquad (3.4)$$

where

$$S_n^* = \frac{1}{2}\sum_{\substack{m=1\\ m\neq n}}^{\infty}\left([(m+n)F_{m+n} - jF_j]\frac{\partial G_m}{\partial x} + m\left[\frac{\partial F_{m+n}}{\partial x} - \mathrm{sgn}(m-n)\frac{\partial F_j}{\partial x}\right]G_m\right)$$

Here, $j = |m-n|$ and $\mathrm{sgn}(m-n)$ denotes the sign of $m-n$, with $\mathrm{sgn}(0) = 0$.

The boundary conditions utilized in conjunction with (3.3) and (3.4) are the transformed conditions (2.10), (2.11) and (2.14) after (3.1) and (3.2) have been applied. This gives

$$F_n = \frac{\partial F_n}{\partial x} = 0 \quad \text{when } x = 0; \quad g(x,\tau) \to 0 \text{ as } x \to \infty \qquad (3.5)$$

and

$$\int_0^{\infty} \mathrm{e}^{(2-n)kx}G_n(x,\tau)\,\mathrm{d}x = 2(\cos\alpha\tau)\delta_{n,1}. \qquad (3.6)$$

With these conditions it is possible to construct an exact solution in powers of τ following the method of Collins & Dennis [7].

If we put $\tau = k = 0$ in (3.3) and (3.4) for the start of the motion, we obtain the same initial solutions

$$G_n(x,0) = 4\pi^{-1/2}\delta_n\,\mathrm{e}^{-x^2}, \qquad F_n = 2[x\,erf\,x - \pi^{-1/2}(1 - \mathrm{e}^{-x^2})]\delta_n, \qquad (3.7)$$

given by Collins & Dennis which satisfy (3.5) and (3.6). From the initial expressions (3.7) we may now build up a perturbation solution in powers of τ following Collins & Dennis [7]. The expansions for the stream function and vorticity can be made in terms of both k and τ. In the first place we may expand ψ and ζ in the form

$$\psi = \psi_0 + k\psi_1 + k^2\psi_2 + \ldots, \qquad \zeta = \zeta_0 + k\zeta_1 + k^2\zeta_2 + \ldots, \qquad (3.8)$$

where $\psi_m \equiv \psi_m(x,\theta,\tau)$, $\zeta_m \equiv \zeta_m(x,\theta,\tau)$. Then each ψ_m, ζ_m is expanded as a series of powers of τ in the form

$$\psi_m(x,\theta,\tau) = \sum_{n=0}^{\infty}\psi_{mn}(x,\theta)\tau^n, \qquad \zeta_m(x,\theta,\tau) = \sum_{n=0}^{\infty}\zeta_{mn}(x,\theta)\tau^n, \qquad (3.9)$$

where each of the coefficients ψ_{mn}, ζ_{mn} consists of combinations of functions of x with periodic functions of θ. The process of derivation of these coefficients follows very closely the procedures described by Collins & Dennis [7]. It is not necessary to give the analysis in detail and we shall give only the expressions which have been derived for a few of the functions in the series in (3.9).

The differential equations for the functions $\psi_{mn}(x,\theta)$, $\zeta_{mn}(x,\theta)$ and the boundary conditions satisfied by these functions can easily be found. In practice each of these functions can be decomposed into a finite set of Fourier-sine components in the coordinate θ with coefficients which are functions of the variable x. In other words, each of the coefficients of the periodic terms in (3.8) and (3.9) may be considered as a function of x, τ and also to be dependent of k. On expansion in powers of τ and k and equating to zero each coefficient of $k^m\tau^n$, we get the conditions which the Fourier-sine components must satisfy. In the following we shall identify only the composite functions.

The leading terms ζ_{00} and ψ_{00} of the expansions are given by means of (3.7), where each term involves the single periodic term $\sin\theta$. The function ζ_{01} involves the single periodic term $\sin 2\theta$. The differential equation which it satisfies and the solution are given by Collins & Dennis [7]. The solution satisfying all the boundary conditions is

$$\begin{aligned}\zeta_{01}(x,\theta) \;=\; & \Big(-4\pi^{-1/2}\left(1+\frac{4}{3}\pi^{-1}\right)(\mathrm{e}^{-x^2}+\pi^{1/2}x\,erf\,x) \\ & +8\left(1+\frac{2}{3}\pi^{-1}\right)x-4x(erf\,x)^2+\frac{4}{3}\pi^{-1}x(3\mathrm{e}^{-X^2}-4)\,\mathrm{e}^{-x^2} \\ & +2\pi^{-1/2}(2x^2-1)\,\mathrm{e}^{-x^2}erf\,x\Big)\sin 2\theta. \qquad (3.10)\end{aligned}$$

and the stream function corresponding to (3.10) is found by integrating $\partial^2\psi/\partial x^2 = \zeta$ twice subject to $\psi = \partial\psi/\partial x = 0$ at $x = 0$.

The term α in (3.6) now enters through ζ_{02}, which satisfies

$$\frac{\partial^2\zeta_{02}}{\partial x^2}+2x\frac{\partial\zeta_{02}}{\partial x}-6\zeta_{02}=r_{021}(x)\sin\theta+r_{023}(x)\sin 3\theta \qquad (3.11)$$

where $r_{021}(x)$ and $r_{023}(x)$ are given in the appendix. The corresponding solutions for ζ_{02} and ψ_{02} consist of a sum of two terms in $\sin\theta$ and $\sin 3\theta$ with coefficients which are functions of x. Exact expressions for these functions have been obtained; because of their long expression we only present

$$\begin{aligned}\lim_{x\to 0}(\zeta_{02}(x,\theta)) \;=\; & \frac{1}{135}\pi^{-5/2}([720\pi^2(2-\alpha^2)-(3402\sqrt{3}-2196)\pi-2816]\sin\theta \\ & +[180\pi^2-(1458\sqrt{3}-1404)\pi+256]\sin 3\theta)\end{aligned}$$

Thus if R is large enough and τ small, the solution is given approximately by

$$\zeta \sim \zeta_{00}+\tau\zeta_{01}+\tau^2\zeta_{02}. \qquad (3.12)$$

The corresponding surface vorticity is given by

$$\begin{aligned}\zeta(0,\theta,\tau) \;\sim\; & \pi^{-1/2}\Big(4+\frac{1}{135}\pi^{-2}[720\pi^2(2-\alpha^2)-(3402\sqrt{3}-2196)\pi-2816]\tau^2\Big) \\ & \times\sin\theta-4\pi^{-1/2}\left(1+\frac{4}{3}\pi^{-1}\right)\tau\sin 2\theta+\frac{1}{135}\pi^{-5/2} \\ & \times[180\pi^2-(1458\sqrt{3}-1404)\pi+256]\tau^2\sin 3\theta. \qquad (3.13)\end{aligned}$$

In order to obtain the terms $\zeta_{1n}(x,\theta)$ of the series for ζ_1, it may be shown that ζ_{1n} satisfies an equation of the type

$$\frac{\partial^2\zeta_{1n}}{\partial x^2}+2x\frac{\partial\zeta_{1n}}{\partial x}-4n\zeta_{1n}=R_{1n}(x,\theta). \qquad (3.14)$$

We find that

$$R_{10}(x,\theta) = 16\pi^{-1/2}x(2x^2-1)\,\mathrm{e}^{-x^2}\sin\theta \tag{3.15}$$

and the solution satisfying all the conditions is

$$\zeta_{10} = ((1-erfx) - 2\pi^{-1/2}x(2x^2+1)\,\mathrm{e}^{-x^2})\sin\theta. \tag{3.16}$$

The stream function is found by integrating the equation

$$\frac{\partial^2\psi_{10}}{\partial x^2} = \zeta_{10} + 2x\zeta_{00}, \tag{3.17}$$

subject to the conditions

$$\psi_{10} = \frac{\partial\psi_{10}}{\partial x} = 0 \quad \text{at } x = 0. \tag{3.18}$$

For the function ζ_{11} we obtain

$$R_{11}(x,\theta) = r_{11}(x)\sin 2\theta, \tag{3.19}$$

where

$$\begin{aligned} r_{11}(x) \;=\; & 4\pi^{-1/2}\,\mathrm{e}^{-x^2}\Big[x(16x^4+6x^2-1)erfx + \left(6+\frac{16}{3}\pi^{-1}\right)x \\ & +2\pi^{-1/2}x^2(8x^2-1)\,\mathrm{e}^{-x^2} - 2x^3\left(1+\frac{28}{3}\pi^{-1/2}x\right)\Big] \\ & -4erfx(1-erfx). \end{aligned} \tag{3.20}$$

The expression (3.20) was not given explicitly by Collins & Dennis [7], but the solution ζ_{11} of (3.14) satisfying all the conditions was given (denoted by ω_{11}) in equation (56), p. 64, with an error of a term $4x^3$ in $L_2(x)$ to be replaced by $4\pi x^3$. The function ζ_{12} satisfies the equation

$$\frac{\partial^2\zeta_{12}}{\partial x^2} + 2x\frac{\partial\zeta_{12}}{\partial x} - 8\zeta_{12} = r_{121}(x)\sin\theta + r_{123}(x)\sin 3\theta \tag{3.21}$$

where $r_{121}(x)$ and $r_{123}(x)$ are given in the appendix. Exact expressions for ζ_{12} and ψ_{12} have been obtained; because of their long expression we only present

$$\begin{aligned} \lim_{x\to 0}(\zeta_{12}(x,\theta)) = & \Bigg(\left(\frac{231}{8}\sqrt{2} - \frac{4785}{256} - \frac{1}{2}\alpha^2\right) - 54\beta\pi^{-1/2} \\ & +\left(\frac{20537}{240} - \frac{2384}{105}\sqrt{2} - \frac{4563}{80}\sqrt{3}\right)\pi^{-1} + \frac{472}{35}\pi^{-2}\Bigg)\sin\theta \\ & +\Bigg(\left(\frac{7607}{256} - \frac{369}{8}\sqrt{2}\right) + 324\beta\pi^{-1/2} \\ & -\left(\frac{1184}{105}\sqrt{2} + \frac{1539}{80}\sqrt{3} + \frac{57991}{1680}\right)\pi^{-1} + \frac{88}{35}\pi^{-2}\Bigg)\sin 3\theta \end{aligned}$$

where β is the integral

$$\beta = \int_0^\infty \mathrm{e}^{-2x^2} erf(x)\,\mathrm{d}x$$

which can be evaluated as

$$\beta = (2\pi)^{-1/2} \arctan(2^{-1/2}). \tag{3.22}$$

The first term ζ_{20} of the series for ζ_2 satisfies the equation

$$\frac{\partial^2 \zeta_{20}}{\partial x^2} + 2x \frac{\partial \zeta_{20}}{\partial x} = -4\pi^{-1/2}(8x^6 - 16x^4 - 1)\,\mathrm{e}^{-x^2} \sin\theta. \tag{3.23}$$

The solution (denoted by ω_{20}) satisfying all the conditions is given as equation (64) by Collins & Dennis [7] and the solution for ψ_{20} is given as equation (65), where a numerical factor 7/24 should be replaced by 1/24. It becomes too complicated to obtain further terms by exact analysis. We thus finally obtain an expression for the vorticity

$$\zeta(x,\theta,\tau) \sim \zeta_{00} + \tau\zeta_{01} + \tau^2\zeta_{02} + k(\zeta_{10} + \tau\zeta_{11} + \tau^2\zeta_{12}) + k^2(\zeta_{20} + O(\tau)) \tag{3.24}$$

which is valid for small τ and large R. In particular, we find for the surface vorticity

$$\begin{aligned}
\zeta(0,\theta,\tau) \sim{}& \left(4\pi^{-1/2} + k - \frac{1}{4}\pi^{-1/2}k^2\right)\sin\theta - \tau \\
&\times\left(\frac{4}{3}\pi^{-3/2}(3\pi+4) - \frac{1}{120}\pi^{-1}(15\pi[96\sqrt{2}-77] - 304)\right)\sin 2\theta \\
&+\tau^2\left(\left(\frac{1}{135}\pi^{-2}[720\pi^2(2-\alpha^2) - (3402\sqrt{3} - 2196)\pi - 2816]\right.\right. \\
&\left.+[1.60170 - \frac{1}{2}\alpha^2]k\right)\sin\theta + \left(\frac{1}{135}\pi^{-5/2}[180\pi^2\right. \\
&\left.\left.-(1458\sqrt{3} - 1404)\pi + 256] + 4.92853k\right)\sin 3\theta\right)
\end{aligned} \tag{3.25}$$

Some further results derived from these solutions will be given in the next section.

4. Results and comparisons

One of the most interesting physical features of the flow is the determination of the time at which the fluid first starts to separate from the cylinder. It occurs at a time $\tau = T$, say, defined by the condition $\partial\zeta/\partial\theta = 0$ for $x = 0$, $\theta = 0$. From the expansion (3.24) in powers of k and τ we can obtain various approximations to T by investigating the roots of the equation

$$\sum_{m=0}^{m_0} \sum_{n=0}^{n_0} k^m T^n (\partial\zeta_{mn}/\partial\theta)_{x=\theta=0} = 0. \tag{4.1}$$

Here m_0 and n_0 correspond to the total number of terms taken in the series (3.12) and (3.24) for $\zeta(x,\theta,\tau)$. The boundary-layer solution corresponds to $m_0 = 0$ and successive approximations to T are obtained by taking increasing values of n_0.

If we take $n_0 = 1$ in the boundary-layer case we obtain

$$T = 3\pi[2(3\pi + 4)]^{-1} \simeq 0.3510217$$

Table 1. Approximations to the time of separation, T, for the boundary layer case ($k = 0$ or $R = \infty$).

α	0	1	2	4	6	8
T	0.319505	0.287028	0.232814	0.157529	0.116830	0.0924374

Table 2. The effect of the first two boundary-layer correction terms on the time of first separation T.

R/α	0	1	2	4	6	8
100	–	–	–	–	–	–
500	0.415	0.344	0.259	0.165	0.120	0.094
10^3	0.378	0.324	0.250	0.163	0.119	0.094
10^4	0.335	0.297	0.238	0.159	0.118	0.093

which is in agreement with the boundary-layer theory. The second approximation ($n_0 = 2$) gives the time T as the positive root of

$$\frac{1}{135}\pi^{-2}\left(4[45\pi^2(11-4\alpha^2) - \pi(1944\sqrt{3} - 2196) - 1602] - 512\right)T^2$$
$$-\frac{8}{3}\pi^{-1}(3\pi + 4)T + 4 = 0. \tag{4.2}$$

This yields the value $T = 0.319504$ when $\alpha = 0$ which is in agreement with the approximation given by Collins & Dennis [7] in the case of an impulsively started circular cylinder without oscillation. Approximations corresponding to values of the Strouhal number have been calculated and these are shown in Table 1. An investigation of the variation of T with R and α has also been carried out by finding the appropriate root of the equation

$$\frac{1}{135}\pi^{-2}(180(11-4\alpha^2)\pi^2 - (7776\sqrt{3} - 6408)\pi - 6920)T^2 + (2\pi T/R)^{1/2}$$
$$\times\left[\frac{1}{3360}\pi^{-2}(105(32\alpha^2 - 7008\sqrt{2} + 4509)\pi^2\right.$$
$$+6168960\beta\pi^{3/2} - (379904\sqrt{2} + 771120\sqrt{3} - 1270928)\pi + 141312)T^2$$
$$\left.+\frac{1}{30}(1440\sqrt{2} - 1150\pi^{-1})T + 2\right] - \frac{8}{3}\pi^{-1}(3\pi + 4)T + 4 = 0 \tag{4.3}$$

incorporating the terms in (5.1) when $m_0 = 1$ and $n_0 = 2$ which have been calculated. The results are shown in Table 2. The results of Table 2 shows that R has a stronger influence on the time of first separation at lower α than at higher α. It is noted that the time of the separation increases with the increase of the steady-streaming Reynolds number $R_s = R/\alpha$ (i.e. with the decrease of the frequency parameter $\mu = R\alpha$) for fixed R. The results of Table 2 verify that the frequency parameters R_s and μ have no effects on the time of the separation in the case of non-oscillating flow when $\alpha = 0$. Also, the effect of increase of Strouhal number is to drastically reduce the time of separation at all Reynolds numbers. In conclusion, if the oscillation amplitude is very small compared with the radius of the cylinder, $\alpha \gg 1$, the flow cannot allow separation.

A dimensionless drag coefficient C_D is defined by $C_D = D/\rho U^2 a$ where D is the total viscous drag on the cylinder. It may be expressed as

$$C_D = \frac{4}{R}\int_0^{\pi}\left(\zeta - \frac{\partial \zeta}{\partial \xi}\right)_{\xi=0} \sin\theta \, \mathrm{d}\theta, \tag{4.4}$$

in which the first term in the integral gives the friction drag coefficient C_f and the second the pressure drag coefficient C_p, where $C_D = C_f + C_p$. Both of these coefficients can be calculated as series in powers of τ and k from the present results. It is found, as far as the terms calculated, that

$$\begin{aligned} C_f = {} & \pi(2R\tau)^{-1/2}\Bigg[4\pi^{-1/2} - \frac{1}{135}\pi^{-5/2}(2[360\pi^2(\alpha^2 - 2) \\ & +9\pi(189\sqrt{3} - 122) + 1408])\tau^2 + k - \left(\frac{1}{4}\pi^{-1/2}\right)k^2 \\ & - \frac{1}{26880}\pi^{-2}(105\pi^2(128\alpha^2 - 7292\sqrt{2} + 4785) + 1451520\beta\pi^{3/2} \\ & +\pi(610304\sqrt{2} + 1533168\sqrt{3} - 2300144) - 362496)k\tau^2 + O(\tau^3)\Bigg] \end{aligned} \tag{4.5}$$

$$\begin{aligned} C_p = {} & -\frac{\pi}{4\tau}\Bigg[8\alpha^2\tau^2 - 4\pi^{-1/2}k - k^2 + \frac{1}{15120}\pi^{-5/2} \\ & \times((80640\pi^2(\alpha^2 - 12\sqrt{2} + 6) + 4354560\beta\pi^{3/2} \\ & -\pi(430080\sqrt{2} - 381024\sqrt{3} - 345408) + 315392)k\tau^2 + O(\tau^3)\Bigg] \end{aligned} \tag{4.6}$$

It may be seen that initially C_f and C_p contribute equally to C_D

$$C_f \simeq C_P \simeq \left(\frac{8\pi}{R\tau}\right)^{1/2} \tag{4.7}$$

in agreement with the result given by Collins & Dennis [7] for the impulsively started circular cylinder. Also, the increase of Strouhal number soon has an appreciable effect on C_D, particularly if R is large.

Finally, we may also obtain an exact expression for the surface pressure coefficient p_0^* defined by

$$p_0^* = \frac{p_0(\theta,\tau) - p_0(\pi,\tau)}{(1/2)\rho U^2} \tag{4.8}$$

valid for small values of τ for viscous flow by making use of (3.24) and

$$p_0^* = -\frac{4}{R}\int_\theta^{\pi}\left(\frac{\partial \zeta}{\partial \xi}\right)_{\xi=0} \mathrm{d}\theta. \tag{4.9}$$

Table 3. Calculated values of the time of separation, T, from the numerical solution.

R/α	0	1	2	4	6	8
100	0.513	0.430	0.315	0.192	0.136	0.105
500	0.394	0.355	0.284	0.183	0.132	0.103
10^3	0.372	0.339	0.276	0.181	0.131	0.102
10^4	0.337	0.313	0.262	0.177	0.130	0.102

This coefficient p_0^* measures the change in dimensionless pressure over the surface of the cylinder starting from the front stagnation point. It may be shown that the expression for p_0^* found from (3.24) using (4.9) is

$$\begin{aligned} p_0^* = \Bigg[& \frac{1}{2\tau} 4(\cos 2\theta - 1)\tau \\ & + \left(8\alpha^2(\cos\theta + 1) - \frac{1}{3}\pi^{-3/2}[\pi(7\sqrt{2} - 60) - 16](\cos 2\theta - 1)\right)\tau^2 \\ & -4\pi^{-1/2}(1 + \cos\theta)k + [(3.009\alpha^2 - 4.59154)(\cos\theta + 1) \\ & -(183.346\alpha^2 - 41.5813)(\cos 3\theta + 1)]k\tau^2 \Bigg] \end{aligned} \tag{4.10}$$

The above physical properties of the initial oscillatory flow have also been calculated using the numerical method of integration given by Badr & Dennis [8]. The solution is started by integrating (3.3) and (3.4) using (3.7) as an initial condition and (3.5) and (3.6) as boundary conditions. An implicit method of integration of Crank-Nicolson type is used, and a given approximation is obtained by truncating the series (2.8) and (2.9). It is not necessary to give the numerical procedure in detail and we shall present only the results of the numerical integrations to check the above analytical expressions at small values of τ. A comparison between the results for ζ obtained from the numerical solutions at small values of τ and obtained from the formula (3.24) is given in fig. 1. The agreement is extremely satisfactory. The frictional and pressure drag coefficients defined in (4.4) have been determined using (4.5) and (4.6) analytically and fig. 2 shows these results together with the ones obtained from the numerical solution for the small values of τ when $R = 10^4$ and $\alpha = 1.0$. There is a remarkable difference between the contributions of the frictional and pressure drag coefficients to the total drag coefficient and C_p has major contribution to C_D where C_f has very little contribution. The analytical results exhibited in fig. 3 compare satisfactorily with results derived from the numerical procedure. As a further check on the consistency of the analytical and numerical results, the pressure coefficient defined by (4.8) is compared for both methods in fig. 3 when $R = 10^4$, $\alpha = 1.0$. The comparison seems satisfactory. We note that the range of R for which the validity of analytical results for T in Table 2 is not known but some evidence can be obtained from the numerical results for T in Table 3.

The vortex formation starts with flow separation at $\theta = 0$ and the growth of such vortex requires time (or distance moved by the fluid). The effects of R and α on the distance moved until separation first occurs can be determined using

$$x/A = \sin(\alpha T) \tag{4.11}$$

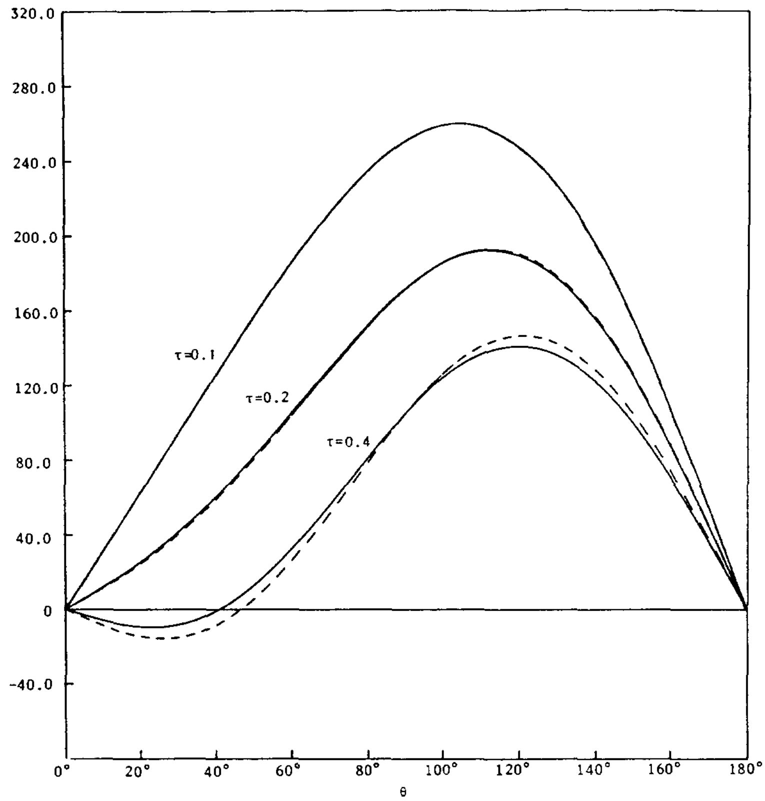

Fig. 1. Comparison of the vorticity distribution over the surface of the cylinder at $R = 10^4, \alpha = 1.0$; —, Numerical Solution; – – –, Analytical Solution.

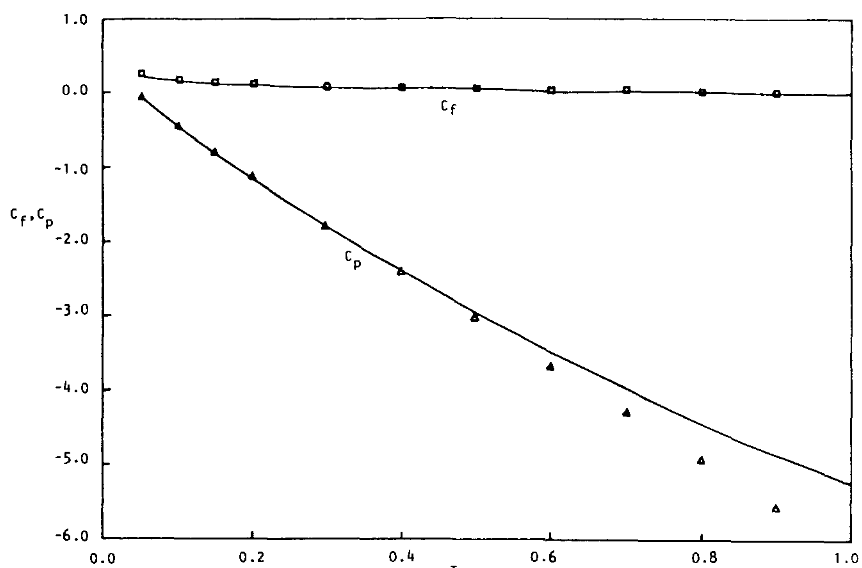

Fig. 2. Variation of C_f and C_p with time at $R = 10^4, \alpha = 1.0$; —, Numerical Solution; □: C_f, △: C_p, Analytical Solution.

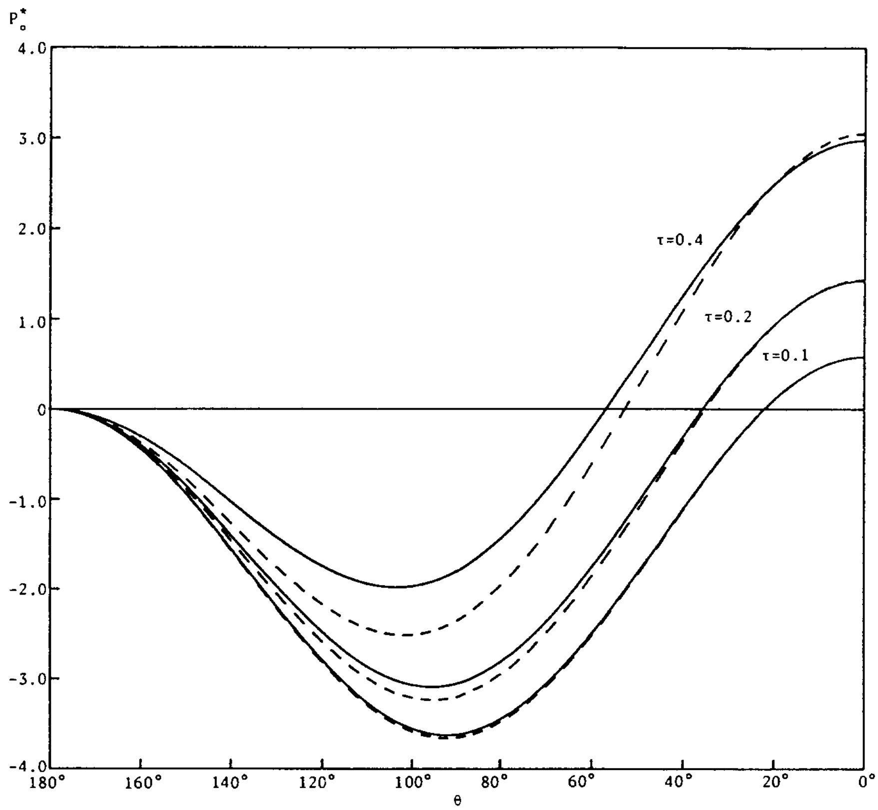

Fig. 3. Comparison of the variation of pressure coefficient over the surface of the cylinder at $R = 10^4$, $\alpha = 1.0$; —, Numerical Solution; - - -, Analytical Solution.

Table 4. Calculated values from the series solution of the distance, x/A, moved until first separation occurs.

R/α	1	2	4	6	8
500	0.338	0.495	0.614	0.661	0.685
10^3	0.318	0.480	0.606	0.656	0.682
10^4	0.293	0.458	0.595	0.648	0.676

Table 5. Calculated values from the numerical solution of the distance, x/A, moved until first separation occurs.

R/α	1	2	4	6	8
500	0.348	0.537	0.669	0.712	0.733
10^3	0.333	0.524	0.662	0.708	0.731
10^4	0.308	0.500	0.651	0.702	0.726

where T is the time of the first separation. The distance x/A in (4.11) is measured as a fraction of the oscillation amplitude A. This distance is determined both analytically and numerically for selected values of R and α and the results are given in the Tables 3 and 4 respectively. These tables show that with the increase of α the relative distance moved, x/A, until separation first occurs mainly depends on α with a very little dependence on R. Also, it seems that at small values of α vortices have sufficient time to grow while for large α the vortices are formed just before the end of stroke leaving very little time for their growth. For example when $\alpha = 0.5$ flow separation starts at a distance less than 1/5 of the amplitude leaving the remaining 4/5 to the vortex formation and growth. On the other hand, when $\alpha = 4.0$, flow separation starts at a distance approximately equal to $2/3$ of the amplitude leaving only 1/3 for the vortex formation and growth.

5. Conclusion

We have determined the initial flow of an oscillating fluid in the presence of a circular cylinder by both an analytical and numerical procedure. The important point about the investigation is that it adopts boundary-layer coordinates and scalings to model the flow but without making any boundary-layer approximations, since the full Navier-Stokes equations are retained for all times. We may note that unless a boundary-layer structure is adopted there is a considerable difficulty in determining the initial flow by numerical methods due to a singularity in the vorticity at $\tau = 0$. In other words, numerical methods which calculate the initial flow in the natural physical coordinates are likely to give inaccurate initial results.

Some trial comparisons between the analytical and numerical results suggest that the analytical expressions are satisfactory at small times and are useful in providing initial details of the flow. These details can be used as starting information to which the numerical methods may be linked.

We have not considered explicitly the question of steady streaming because the time taken for this to occur would be well beyond the validity of the present results. Major numerical calculations are required to study this phenomenon for longer times.

Appendix

$$\begin{aligned}
r_{021} = {} & -\frac{16}{9}\pi^{-2}(6\sqrt{\pi}(3\pi+2) - 2[3(4x^4 - 3x^2 + 3)\pi^{3/2} + 6\pi x \\
& +\sqrt{\pi}(8x^4 - 6x^2 + 9) + 8x]\,\mathrm{e}^{-x^2} + [(3\pi+4)x(4x^2+1) \\
& -6\sqrt{\pi}(7x^2 - 4)]\,\mathrm{e}^{-2x^2} + \sqrt{\pi}(5x^2 - 3)\,\mathrm{e}^{-3x^2}) - 48x(erf x)^3 \\
& -\frac{8}{3\pi}(6[3\pi x - \sqrt{\pi} + 4x] + \sqrt{\pi}(20x^4 - 24x^2 + 15)\,\mathrm{e}^{-x^2}(erf x)^2 \\
& +\frac{8}{9}\pi^{-3/2}(6[18x\pi^{3/2} + 3\pi + 12\sqrt{\pi}x + 4] \\
& +[(3\pi+4)(8x^4 + 6x^2 + 15) - 6\sqrt{\pi}x(14x^2 - 1)]\,\mathrm{e}^{-x^2} \\
& -12\sqrt{\pi}x(10x^2 - 17)\,\mathrm{e}^{-2x^2})(erf x) - \frac{256}{3\pi}\sqrt{2}x(erf\sqrt{2}x)\,\mathrm{e}^{-x^2}
\end{aligned}$$

$$r_{023}(x) = -\frac{16}{9}\pi^{-2}(6\sqrt{\pi}(3\pi+2) - 2[3(4x^4 - 3x^2 + 3)\pi^{3/2} + 6\pi x$$

$$+\sqrt{\pi}(8x^4 - 30x^2 - 9) + 8x]\,e^{-x^2} - [(3\pi + 4)x(4x^2 + 1)$$
$$+6\sqrt{\pi}(7x^2 - 4)]\,e^{-2x^2} + 6\sqrt{\pi}(x^2 - 3)]\,e^{-3x^2} + 16x(erf x)^3$$
$$+\frac{8}{3\pi}(2[3\pi x - \sqrt{\pi} + 4x] - \sqrt{\pi}(4x^4 + 24x^2 + 3)\,e^{-x^2})(erf x)^2$$
$$-\frac{8}{9}\pi^{-3/2}(6[6x\pi^{3/2} - 3\pi + 4\sqrt{\pi}x - 4]$$
$$+[(3\pi + 4)(8x^4 + 18x^2) + 9 + 6\sqrt{\pi}x(14x^2 - 1)]\,e^{-x^2}$$
$$+12\sqrt{\pi}x(2x^2 + 11)\,e^{-2x^2})(erf x) + \frac{256}{3\pi}\sqrt{2}x(erf\sqrt{2}x)\,e^{-x^2}$$

$$r_{121}(x) = \frac{1}{90}\pi^{-5/2}\Big(\sqrt{\pi}[2880\pi^2x^2(2\alpha^2 + 1) + x\pi^{3/2}(28260 - 17280\sqrt{2})$$
$$+48\pi(392x^2 - 5) + 12480\sqrt{\pi}x + 22528x^2]$$
$$-(\pi^2 x[5760x^6 - x^4(5760\sqrt{2} + 6900) - x^2(11520\sqrt{2} + 1800)$$
$$+5760\alpha^2 - 8640\sqrt{2} + 10170] + \pi^{3/2}[2016x^4$$
$$+x^2(15576 - 23040\sqrt{2}) - 1150\sqrt{2} + 12480]$$
$$+16\pi x(240x^6 - 420x^4 - 628x^2 + 1701\sqrt{3} + 1626)$$
$$+128\sqrt{\pi}(36x^4 + x^2 + 40) + 22528x)\,e^{-x^2}$$
$$+2\sqrt{\pi}(\pi[1440x^6 + x^4(270 - 2880\sqrt{2}) + x^2(405 - 4320\sqrt{2})$$
$$-5760\sqrt{2} + 6360] - 48\sqrt{\pi}x(110x^4 - 37x^2 - 294)$$
$$+80(24x^6 - 14x^4 - 63x^2 + 32))\,e^{-2x^2}$$
$$+96\pi x(75x^4 - 71x^2 + 44)\,e^{-3X^2} + 4(4x^2 + 1)(erf x)^3$$
$$-\frac{1}{3}\pi^{-1}(\pi[x^2(1152\sqrt{2} - 732) + 288\sqrt{2} - 225] + 228\sqrt{\pi}x$$
$$-16(8x^2 + 3) - 4\sqrt{\pi}x(60x^6 - 112x^4 - 183x^2 - 6)\,e^{-x^2})$$
$$\times(erf x)^2 + \frac{1}{90}\pi^{-2}(90\pi^2[4x^2(16\alpha^2 - 96\sqrt{2} + 73) - 96\sqrt{2} + 79]$$
$$+x\pi^{3/2}(19620 - 17280\sqrt{2}) + 48\pi[x^2(567\sqrt{3} + 472) + 25]$$
$$+12480\sqrt{\pi}x + 22528x^2 - \sqrt{\pi}(15\pi x[192x^2 + x^4(132 - 384\sqrt{2})$$
$$-x^2(768\sqrt{2} + 24) - 3168\sqrt{2} + 2433] - 48\sqrt{\pi}(220x^6 + 36x^4$$
$$-575x^2 - 42) + 80x(48x^6 - 4x^4 - 164x^2 + 123))\,e^{-x^2}$$
$$-480\pi(30x^6 - 71x^4 - 71x^2))(erf x) + \frac{64}{15}\sqrt{2}\pi^{-1}$$
$$\times(36x^4 - 24x^2 - 5)\,e^{-X^2}(erf(\sqrt{2}x))$$
$$+\frac{1512}{5}\sqrt{3}x^2(erf(\sqrt{3}x))\Big)$$

$$r_{123}(x) = \frac{1}{180}\pi^{-5/2}\Big(8\sqrt{\pi}(720\pi^2x^2 + x\pi^{-3/2}(4320\sqrt{2} - 4185)$$

$$
\begin{aligned}
&+12\pi(72x^2+5)-1200\sqrt{\pi}x+512x^2-2(\pi^2 x[5760x^6\\
&+x^4(2700-5760\sqrt{2})+x^2(5520-23040\sqrt{2}))\\
&+8640\sqrt{2}-5850]+\pi^{3/2}[4896x^4+x^2(22056-23040\sqrt{2})\\
&+1150\sqrt{2}-8640]+48\pi x(80x^6+260x^4-84x^2\\
&-243\sqrt{3}-146)+128\sqrt{\pi}x^2(36x^2+71)+2048x)\,\mathrm{e}^{-x^2}\\
&+\sqrt{\pi}(\pi[5760x^6+x^4(15480-11520\sqrt{2})\\
&+x^2(29220-4320\sqrt{2})+23040\sqrt{2}-17760]\\
&+192\sqrt{\pi}x(110x^4-37x^2+102)+320x^2(24x^4\\
&+26x^2-7))\,\mathrm{e}^{-2x^2}-576\pi x(5x^4-13x^2+92)\,\mathrm{e}^{-3x^2}\\
&-4(12x^2-1)(erf\,x)^3+\frac{1}{3}\pi^{-1}(3\pi[32x^2+96\sqrt{2}-79]-96\sqrt{\pi}x\\
&+16(4x^2-3)+12\sqrt{\pi}x(4x^6+32x^4+39x^2-30)\,\mathrm{e}^{-x^2})\\
&\times(erf\,x)^2+\frac{1}{90}\pi^{-2}(270\pi^2(16x^2-32\sqrt{2}+25)\\
&+x\pi^{3/2}(17280\sqrt{2}-13860)-48\pi[x^2(243\sqrt{3}-32)-35]\\
&-4800\sqrt{\pi}x+2048x^2-\sqrt{\pi}(15\pi x[192x^6+x^4(612-384\sqrt{2})\\
&+x^2(1136-1536\sqrt{2})+288\sqrt{2}-315]+48\sqrt{\pi}(220x^6+36x^4\\
&+217x^2+42)+80x(48x^6+76x^4-12x^2+3))\,\mathrm{e}^{-x^2}\\
&+1440\pi(2x^6+15x^4+11x^2-6))(erf\,x)\\
&-\frac{64}{5}\sqrt{2}\pi^{-1}(12x^4-8x^2-5)\,\mathrm{e}^{-X^2}(erf(\sqrt{2}x))\\
&+\frac{648}{5}\sqrt{3}x^2(erf(\sqrt{3}x))\Big)
\end{aligned}
$$

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